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Kelvin–Helmholtz instability of a slowly varying flow

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The linear stability of a basic flow of two homogeneous inviscid incompressible fluids under the action of gravity is treated mathematically. In the basic state, one fluid is at rest below a horizontal plane $z = 0$; and the other flows above in the x direction, its speed varying slowly with the lateral co-ordinate y. The eigenvalue problem for normal modes is derived; its equation is a partial differential one, the co-ordinates y and z not being separable. The problem is solved approximately by taking the modes *locally* as if the basic velocity were independent of y, though the lateral wavenumber is allowed to vary slowly with y. This leads to an ordinary differential equation in y which is solved by the JWKB method. Detailed calculations are made for a parabolic profile, representing the blowing of air over water in a wide channel, and for other profiles.

"There is nothing Nature loves so well as to change existing forms and to make new ones like them." Marcus Aurelius, *Meditations IV*

1. Introduction

The problem to be posed and treated below is a model of instability when air is blown over water in a wide long channel. More generally it may be regarded as a prototype of Kelvin–Helmholtz instability of flows which vary slowly. Such problems are important in many applications, because it is rare in practice that a flow both is steady and depends upon only one space co-ordinate. We shall treat the hydrodynamic instability of inviscid incompressible homogeneous fluids bounded by rigid side walls at $y = Y_1/\epsilon$ and Y_2/ϵ, the basic horizontal velocity and density being given by

$$\mathbf{U} = \begin{cases} U(\epsilon y)\,\mathbf{i}, \\ 0, \end{cases} \quad \rho = \begin{cases} \rho_0(1-\delta) & \text{for} \quad z > 0, \\ \rho_0(1+\delta) & \text{for} \quad z < 0, \end{cases} \tag{1}$$

respectively, where z is the height, Y_1 and Y_2 are fixed lengths and ϵ is a small positive dimensionless parameter. The gravitational acceleration will be denoted by $-g\mathbf{k}$, and ρ_0 and δ taken as positive parameters with $\delta < 1$.

Motivated by various problems of geophysical fluid dynamics, Blumen (1971, 1973, 1974) has considered various general and particular characteristics of similar three-dimensional hydrodynamic instability. Barcilon & Drazin (1972) applied a similar model to the formation of a dust devil. These papers

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treated three-dimensional Kelvin–Helmholtz instability without slow variation of the basic flow. Longuet-Higgins & Stewart (1961) considered slow variation without instability, treating the propagation of short waves on an ocean with basic shear. Here we shall treat the above problem of three-dimensional Kelvin–Helmholtz instability with slow variation by a method applied by Drazin (1974) to a model problem. Gent (1974) has also applied the method to problems of baroclinic instability. A somewhat similar method has been applied by Bouthier (1972, 1973) to the stability of a boundary layer on a plate.

The general eigenvalue problem for the stability of the basic state (1) is posed in §2. The approximation of slow lateral variation is introduced in §3, and found to lead to a JWKB eigenvalue problem. This problem is solved fairly generally in §4. Detailed stability characteristics are found in §5 for a symmetric parabolic velocity profile. A piecewise-constant velocity profile is used in §6 to elucidate unstable modes. Finally, stability characteristics of a linear profile are found in §7.

2. The general problem

The equations of motion and incompressibility apply on each side of the disturbed interface, say $z = \zeta(x, y, t)$, between the two fluids. We express the total velocity $\mathbf{U} + \mathbf{u}'$ and the total pressure $p_0 - g\rho z + p'$ each as the sum of the basic quantity and its small perturbation, and then linearize the equations in the usual way by neglecting products of ζ and the other small quantities $\mathbf{u}'$ and p'. By the method of normal modes we also suppose that $\mathbf{u}'$, p', $\zeta \propto \exp\{ik(x-ct)\}$ for a given wavenumber k, and seek eigenvalues c for each basic flow and thence the stability characteristics of that flow. Then the equations become

$$ik(U-c)\,u' + ikp'/\rho = -U_y v', \tag{2}$$

$$ik(U-c)\,v' + p'_y/\rho = 0, \tag{3}$$

$$ik(U-c)\,w' + p'_z/\rho = 0, \tag{4}$$

$$iku' + v'_y + w'_z = 0. \tag{5}$$

It is now possible to differentiate these equations and eliminate all the dependent variables but v' to find that v' satisfies the Rayleigh stability equation.

So far we have merely summarized what is given in any standard treatment of instability of parallel flow (cf. Drazin & Howard 1966, §2.1). However, here we must take account of the two layers of fluid and exploit the slowness of the variation of U with y. Accordingly, we define the 'slow' variable by $Y = \epsilon y$, and eliminate all the variables but w' from (2)–(5). This can be done quite simply to give the equation of motion for each fluid (with $U \equiv 0$ in the lower fluid):

$$(U-c)\,(w'_{yy} + w'_{zz} - k^2 w') = \epsilon^2 \left(\frac{2U_Y^2}{U-c} - U_{YY}\right) w'. \tag{6}$$

Linearization of the kinematic boundary condition at the interface $z = \zeta$ between the fluids of different densities gives

$$[w'/(U-c)] = 0 \quad \text{at} \quad z = 0, \tag{7}$$

the square brackets denoting the difference between their contents on the two sides of the horizontal plane $z = 0$. Continuity of the pressure at the interface leads to the condition

$$\left[\rho\left\{(U-c)\,w'_z+\frac{g}{k^2}\frac{w'_{yy}-k^2w'}{U-c}-2\epsilon U_Y v'+\frac{2\epsilon g}{k^2}\frac{U_Y w'_y}{(U-c)^2}+\frac{\epsilon^2 g}{k^2}\frac{(U-c)\,U_{YY}-U_Y^2}{(U-c)^3}w'\right\}\right]=0 \quad \text{at} \quad z=0, \qquad (8)$$

after some differentiation and partial elimination. The condition that the normal velocity vanishes at the side walls leads to the conditions

$$w'_y = -\frac{\epsilon U_Y}{U-c}w' \quad \text{at} \quad Y = Y_1, Y_2. \qquad (9)$$

Finally, we require that

$$w' \quad \text{is bounded as} \quad z \to \pm\infty. \qquad (10)$$

Equation (6), boundary conditions (7)–(10) and the x component of the vorticity equation (which enables one to eliminate v' from condition (8)) constitute the eigenvalue problem of finding the stability characteristics for general values of ϵ.

Before solving the problem for small ϵ, we shall make two general remarks. First, we may take $k \geqslant 0$ without loss of generality because there is symmetry between k and $-k$ (cf. Drazin & Howard 1966, p. 9). Second, generalization of Howard's (1961) semicircle theorem for this problem shows (Blumen 1974) that if there is instability then the eigenvalue $c = c_r + ic_i$ lies in the semicircle

$$\{c_r - \tfrac{1}{2}(U_{\max}+U_{\min})\}^2 + c_i^2 \leqslant \{\tfrac{1}{2}(U_{\max}-U_{\min})\}^2, \quad c_i > 0, \qquad (11)$$

in the complex c plane provided that δ is neglected, though $g\delta$ need not be. In fact this result is also valid for constant U when δ is not negligible [see equation (16)] and may be so for variable U. The result proves helpful in computing complex values of c.

3. The problem with slow variation

When ϵ is small we may use the local approximation that at each value of Y the form of a normal mode is as if the basic flow were independent of Y although the local wavenumber l in the y direction may vary with Y. This approximation, the heart of the method of this paper, is justified at length by Bouthier (1972) and by Drazin (1974), who acknowledge others who developed the method for slowly varying waves. This leads here to an asymptotic solution of the form

$$w' \sim \begin{Bmatrix} W_+(z,Y) \\ W_-(z,Y) \end{Bmatrix} f(y,\epsilon)\, e^{ik(x-ct)} \quad \text{for} \quad \begin{Bmatrix} z > 0 \\ z < 0 \end{Bmatrix} \qquad (12)$$

as $\epsilon \to 0$. This form will lead to a normal mode approximately satisfying the equations of motion and boundary conditions in z for each value of Y. However, the perturbation f must 'fit' between the side walls with a definite number of zeros, much as the wave function of a bound state does in quantum mechanics.

This is effected by the slow variation of the y wavelength as in the JWKB approximation. In other words, the solution will be found to approximate a local solution at each value of Y, but the wavelength will vary slowly with Y and the complex velocity c will be determined globally. To see how these results are found, note that, to zeroth order in ϵ, equation (6) and conditions (7) and (10) give

$$f_{yy} + l^2 f = 0 \tag{13}$$

and
$$W_+ = (U-c)\exp\{-(k^2+l^2)^{\frac{1}{2}}z\}, \quad W_- = -c\exp\{(k^2+l^2)^{\frac{1}{2}}z\}, \tag{14}$$

where $\mathrm{Re}\,(k^2+l^2)^{\frac{1}{2}} \geqslant 0$ and $l^2(Y,c)$ may be a complex function.

In the conventional problem of Kelvin–Helmholtz instability that would arise if U were independent of Y, one would take l as a given wavenumber in the y direction and use the dynamic condition (8) to find the eigenvalue c for given real parameters k and l. However, here we satisfy condition (8) as $\epsilon \to 0$ in order to find the *function* $l^2(Y,c)$ and thereafter find the *constant* complex eigenvalue c by solving (13) with the side-wall conditions (9).

First, as $\epsilon \to 0$, condition (8) gives us

$$l^2(Y,c) = (k^2/g\delta)^2(\{c-\tfrac{1}{2}(1-\delta)\,U\}^2 + \tfrac{1}{4}(1-\delta^2)\,U^2)^2 - k^2 \tag{15}$$

or
$$c = \tfrac{1}{2}(1-\delta)\,U \pm \{g\delta(k^2+l^2)^{\frac{1}{2}}/k^2 - \tfrac{1}{4}(1-\delta^2)\,U^2\}^{\frac{1}{2}}. \tag{16}$$

Equation (16) would be Kelvin's eigenvalue relation if U were independent of Y, but here we use (15) to determine l^2 as a function of Y and c, although the eigenvalue c has yet to be determined.

To determine c we need (13) and the side-wall conditions (9) as $\epsilon \to 0$. These give

$$f_{YY} + \epsilon^{-2} l^2 f = 0, \tag{17}$$

$$f_Y = 0 \quad \text{at} \quad Y = Y_1, Y_2. \tag{18}$$

Now the eigenvalue problem (17), (18) and (15) involves an ordinary differential equation and is suitable to be solved by the JWKB method as $\epsilon \to 0$.

Note that the assumption of the form (12) has enabled us to satisfy all the equations and boundary conditions to the first approximation for small ϵ. Thus, the next approximation would lead to terms of order ϵ^{-1} on the right-hand side of (17) and of order ϵ on the right-hand side of (18). Therefore it is consistent to use only the first JWKB approximation, called the semi-classical approximation by some physicists.

It is convenient to change to dimensionless variables here. So choose D as some positive length scale and V as a velocity scale of the basic velocity profile. Then we may use V and D to scale all dimensional variables and parameters in the usual way to derive a dimensionless problem. In this way the form of (17) and (18) is unchanged, but (15) gives

$$l^2 = k^2F^4(\{c-\tfrac{1}{2}(1-\delta)\,U\}^2 + \tfrac{1}{4}(1-\delta^2)\,U^2)^2 - k^2, \tag{19}$$

where the positive Froude number F of the modes of wavenumber k is defined by

$$F \equiv (kV^2/g\delta D)^{\frac{1}{2}}. \tag{20}$$

Now (16) may be used to give the *local* criterion of stability ($c_i \leqslant 0$) of a mode of given wavenumber k as

$$U^2 \leqslant 4/(1-\delta^2)F^2 \tag{21}$$

for each value of Y. To the crudest approximation, one might anticipate that if this criterion were satisfied everywhere the flow would be stable, and if it were violated somewhere the flow would be unstable. This must apply to short waves which are essentially dependent only upon behaviour of the flow in a small neighbourhood, but it so happens that the basic flow (1) is most unstable to long waves. Our analysis confirms the anticipated local criterion in part, but reveals more complications. Certainly, the primary effect of the variation of U with Y is the local one of varying the velocity difference of the two fluids at their interface, but the lateral wavenumber l varies between the side walls as a consequence of the variable velocity difference. We shall find Kelvin–Helmholtz instability in a vertical plane with no possibility of inflexional instability due to the slow variation of the basic velocity in a horizontal plane, but local ideas alone are inadequate to interpret the detailed stability characteristics.

4. The JWKB problem

The system (17)–(19) is an unusual JWKB eigenvalue problem in the sense that the eigenvalue c is not simply a factor of l^2. It seems easiest to solve this problem by inverting it so that we find the lateral wavenumber in terms of c rather than vice versa. So we shall assume that c has a given value, or at any rate lies in some given region of the complex plane, and finally check the consistency of the assumption. The logical basis of this argument is the construction of a complete approximate solution so that c is known for all wavenumbers.

4.1. *Stability*

First suppose that the mode is globally stable. Therefore, c is real and (19) gives real l^2. Then (17) and (18) are a classical real JWKB problem, whose solution depends crucially upon the nature of the zeros of $l^2(Y,c)$ along the real-Y axis between Y_1 and Y_2 (cf. Jeffreys 1962, §3.6).

(i) If $l^2 > 0$ in the interval (Y_1, Y_2), then the eigenfunction is given by

$$f \sim l^{-\frac{1}{2}} \cos\left\{\epsilon^{-1}\int_{Y_1}^{Y} l\, dY\right\} \tag{22}$$

and the eigenvalue relation by

$$\epsilon^{-1}\int_{Y_1}^{Y_2} l\, dY \sim m\pi \quad \text{as} \quad m \to \infty, \tag{23}$$

where the positive integer m is the number of modes of the solution (22) between the side walls at $Y = Y_1, Y_2$, and we take $\operatorname{Re} l > 0$. The JWKB approximation is properly valid only for large values of m, but when $U(Y)$ is a smooth function, one may expect fair quantitative agreement for values of m that are not large (cf. Drazin 1974). We may regard the average wavenumber $m\pi\epsilon/(Y_2 - Y_1)$ as fixed while $\epsilon \to 0$ or $m \to \infty$.

If U were constant, relation (23) would always arise, even if c were complex, and give the classical eigenvalues (16) on identification of $m\pi\epsilon/(Y_2-Y_1)$ with l. However, here U varies with Y, so relation (23) does not give c explicitly in terms of the wavenumbers k and m, but rather gives m in terms of c and the other parameters. It is for this reason we have sought to solve the inverse problem.

In fact $l^2 > 0$ for real c if and only if either $F^{-2} < \frac{1}{4}(1-\delta^2)\,U^2$ or

$$|\,c-\tfrac{1}{2}(1-\delta)\,U| > \{F^{-2}-\tfrac{1}{4}(1-\delta^2)\,U^2\}^{\frac{1}{2}}$$

at a given value of Y, i.e. either the mode is locally unstable or it is locally stable and its speed is greater than that of an internal gravity wave relative to the mean flow. In this case, with $l^2 > 0$ throughout the interval (Y_1, Y_2) of flow, it can be seen that the stability characteristics are not dissimilar from those with constant U, the eigenvalue c being in some sense a Y average of the eigenvalues for local velocity differences $U(Y)$.

(ii) If l^2 has one simple zero Y', such that

$$l^2(Y) > 0 \quad \text{for} \quad Y_1 \leqslant Y < Y', \quad l^2(Y) < 0 \quad \text{for} \quad Y' < Y \leqslant Y_2, \tag{24}$$

say, then the JWKB method can be readily shown (cf. Jeffreys 1962, §3.6) to give the eigenfunction

$$f' \sim \begin{cases} l^{-\frac{1}{2}}\sin\left(\epsilon^{-1}\displaystyle\int_Y^{Y'} l\,dY + \tfrac{1}{4}\pi\right) & \text{for} \quad Y_1 \leqslant Y < Y', \\ |l|^{-\frac{1}{2}}\exp\left(-\epsilon^{-1}\displaystyle\int_{Y'}^{Y_2}|l|\,dY\right)\cosh\left(\epsilon^{-1}\displaystyle\int_Y^{Y_2}|l|\,dY\right) & \text{for} \quad Y' < Y \leqslant Y_2, \end{cases} \tag{25}$$

and the eigenvalue relation

$$\epsilon^{-1}\int_{Y_1}^{Y'} l\,dY = (m-\tfrac{3}{4})\,\pi + o(1) \quad \text{as} \quad m \to \infty.$$

Although we have included the phase $\frac{3}{4}\pi$ to facilitate reference to texts on the JWKB method, our method is not so accurate, as discussed in §3. So, more properly, we shall use the relation

$$\epsilon^{-1}\int_{Y_1}^{Y'} l\,dY \sim m\pi \quad \text{as} \quad m \to \infty. \tag{26}$$

(iii) If l^2 has two simple zeros Y' and Y'' such that

$$l^2(Y) < 0 \quad \text{for} \quad Y_1 \leqslant Y < Y', \quad Y'' < Y \leqslant Y_2; \quad l^2(Y) > 0 \quad \text{for} \quad Y' < Y < Y'', \tag{27}$$

then (cf. Jeffreys 1962, §3.61) the eigenfunction is

$$f \sim \begin{cases} |l|^{-\frac{1}{2}}\exp\left(-\epsilon^{-1}\displaystyle\int_{Y_1}^{Y'}|l|\,dY\right)\cosh\left(\epsilon^{-1}\displaystyle\int_{Y_1}^{Y}|l|\,dY\right) & \text{for} \quad Y_1 \leqslant Y < Y', \\ l^{-\frac{1}{2}}\sin\left(\epsilon^{-1}\displaystyle\int_Y^{Y''} l\,dY + \tfrac{1}{4}\pi\right) & \text{for} \quad Y' < Y < Y'', \\ |l|^{-\frac{1}{2}}\exp\left(-\epsilon^{-1}\displaystyle\int_{Y''}^{Y_2}|l|\,dY\right)\cosh\left(\epsilon^{-1}\displaystyle\int_Y^{Y_2}|l|\,dY\right) & \text{for} \quad Y'' < Y \leqslant Y_2, \end{cases} \tag{28}$$

and the eigenvalue relation

$$\epsilon^{-1}\int_{Y'}^{Y''} l\,dY \sim m\pi \quad \text{as} \quad m \to \infty. \tag{29}$$

(iv) If l^2 has two simple zeros Y' and Y'' such that

$$l^2(Y) > 0 \quad \text{for} \quad Y_1 \leqslant Y < Y', \quad Y'' < Y \leqslant Y_2; \quad l^2(Y) < 0 \quad \text{for} \quad Y' < Y < Y'' \tag{30}$$

then either the eigenfunction is

$$f \sim \begin{cases} l^{-\frac{1}{2}}\sin\left(\epsilon^{-1}\int_{Y}^{Y'} l\,dY + \frac{1}{4}\pi\right) & \text{for} \quad Y_1 \leqslant Y < Y', \\ \frac{1}{2}|l|^{-\frac{1}{2}}\exp\left(-\epsilon^{-1}\int_{Y'}^{Y} l\,dY\right) & \text{for} \quad Y' < Y < Y'', \\ \dfrac{\frac{1}{2}l^{-\frac{1}{2}}\exp\left(-\epsilon^{-1}\int_{Y'}^{Y''} l\,dY\right)\cos\left(\epsilon^{-1}\int_{Y}^{Y_2} l\,dY\right)}{\cos\left(\epsilon^{-1}\int_{Y''}^{Y_2} l\,dY + \frac{1}{4}\pi\right)} & \text{for} \quad Y'' < Y \leqslant Y_2, \end{cases} \tag{31}$$

with the eigenvalue relation

$$\epsilon^{-1}\int_{Y_1}^{Y'} l\,dY = (n - \tfrac{3}{4})\pi + o(1) \quad \text{as} \quad n \to \infty \tag{32}$$

through integral values, or the eigenfunction and relation have similar forms with the roles of Y_1 and Y_2 reversed.

Note that approximation (31) breaks down when

$$\cos\left(\epsilon^{-1}\int_{Y''}^{Y_2} l\,dY + \tfrac{1}{4}\pi\right) = 0,$$

and so, in particular, when l is an even function of Y and $Y_1 = -Y_2$. Also, the present subcase (iv) is more complicated than the previous ones, the two intervals of flow where $l^2 > 0$ being essentially isolated from one another by the interval where $l^2 < 0$. Although n is an integer in this subcase, it is no longer the number m of nodes of the mode between the side walls; nonetheless, it is possible to order the *union* of the two sets of eigenfunctions [(31) and the corresponding set with (Y_1, Y') replaced by (Y_2, Y'')] by the number of zeros of each eigenfunction. As one changes from subcase (i) to subcase (v), the number of zeros will change continuously. These remarks are amplified in some of the examples in later sections.

(v) Further special cases arise according to the nature of the subintervals of (Y_1, Y_2) over which l^2 is either positive or negative. However, their details proliferate and few seem likely to occur in practice, so we shall not examine them here.

In summary of subcases (i)–(v), we see that real eigenvalues c may be used to calculate the y wavenumber $m\pi\epsilon/(Y_2 - Y_1)$ for a given profile $U(Y)$ and given values of F, k, δ and ϵ. If all positive wavenumbers can be found in this way, we conclude that the flow is stable. Otherwise the remaining range of y wavenumbers gives instability, although the growth rates of unstable disturbances can be found only by use of the JWKB approximation in the complex plane, as in §4.2.

However, we shall first amplify the remark at the end of §3 by noting that although $U(Y)$ may equal c for some real value of Y in the interval (Y_1, Y_2) of flow there is no singularity in a critical layer like that of the Rayleigh stability equation. This is because instability due to inflexion points in the x, y plane is not being considered to the present order of approximation for small ϵ. Kelvin–Helmholtz instability in the z, x plane is being perturbed, and the critical layer in that plane occurs always at the interface between the two fluids.

Finally, note that in each subcase the stable disturbances are exponentially small in those regions of space where the local wavenumber l is imaginary and oscillate sinusoidally where l is real.

4.2. *Instability*

Although the criterion for stability can be found by considering only stable disturbances, we shall examine unstable ones briefly. For a mode that is generally unstable, c is expected to lie within the semicircle (11). The JWKB problem (17)–(19) is now complex, and not covered well in the literature. However, Heading (1962, §5.4) and Fröman & Fröman (1965, chap. 7) have treated the complex problem briefly.

It may help to bear in mind the general idea that (17) has two independent exact solutions which are integral functions of Y for each value of ϵ, however small, although the two JWKB approximate solutions [e.g. (22)] have infinities and branch points at the transition points in the complex Y plane where l^2 vanishes. Consequently, different linear combinations of the two JWKB solutions may approximate the same exact solution in different parts of the complex plane, this being the Stokes phenomenon. To apply boundary conditions at Y_1 and Y_2 we seek the relationships between linear combinations of the approximate solutions at these points. These relationships can be found if we take the integrals $\int l dY$ that arise in the exponents of the approximate solutions along a contour in the complex Y plane that neither crosses a cut of the function $l^{\frac{1}{2}}$ nor goes near a zero of l.

This having been said, one has first to find the zeros of l in the complex Y plane, then choose cuts and branches of $l^{\frac{1}{2}}$ appropriately to define the JWKB approximate solutions. This leads to eigenvalue relations similar in form to the ones in §4.1 above, but the contours of integration are along the anti-Stokes lines in the complex Y plane (lines along which the integral $\int l dY$ is real) between the appropriate transition points and along lines joining the end-points Y_1 and Y_2 to infinity.

5. Parabolic profile

To represent wind blowing over water in a wide high channel, take

$$U = 1 - Y^2 \quad \text{for} \quad -1 \leqslant Y \leqslant 1. \tag{33}$$

The problem (17)–(19) is then symmetric in Y about $Y = 0$, so each eigenfunction is either an even or an odd function of Y. Also criterion (21) gives local stability where

$$(1 - Y^2)^2 \leqslant 4/F^2(1 - \delta^2)$$

and thus local stability everywhere if

$$F \leqslant 2/(1-\delta^2)^{\frac{1}{2}}. \tag{34}$$

Note that F is proportional to the square root of k, so that short waves are always locally unstable in this simple model without surface tension or dissipation.

(*a*) First suppose that the values of K, m, F, δ and ϵ are such that c is real. Then the various subcases can be classified, after use of a lot of elementary algebra, as follows.

(i) If either $|c| \geqslant F^{-1}\{2/(1+\delta)\}^{\frac{1}{2}}$ or if none of the subcases (iii), (iv) and (v) arise, then $l^2(Y) > 0$ over the whole interval $(-1, 1)$ of flow, and eigenvalue relation (23) can be shown to give

$$\int_0^1 l\,dY \sim \tfrac{1}{2}m\pi\epsilon \quad \text{as} \quad m \to \infty. \tag{35}$$

(iii) If $-F^{-1} < c < F^{-1}$ and $c < 1$ and either (I) $F^{-1} < \frac{1}{2}(1-\delta^2)^{\frac{1}{2}}$ or

(II) $F^{-1} > \frac{1}{2}(1-\delta^2)^{\frac{1}{2}}$ and $|c-\frac{1}{2}(1-\delta)| > \{F^{-2}-\frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}$,

then $l^2(Y) > 0$ over the subinterval $(-Y', Y')$ and $l^2(Y) < 0$ elsewhere over the interval of flow, where the positive zeros of l^2 are defined by

$$Y', Y'' = +(1-c \mp [\{2F^{-2}-(1+\delta)c^2\}/(1-\delta)]^{\frac{1}{2}})^{\frac{1}{2}} \tag{36}$$

respectively. Then eigenvalue relation (29) gives

$$\int_0^{Y'} l\,dY \sim \tfrac{1}{2}m\pi\epsilon \quad \text{as} \quad m \to \infty. \tag{37}$$

(iv) If $F^{-1} < c < F^{-1}\{2/(1+\delta)\}^{\frac{1}{2}}$ and $F^{-1} > \frac{1}{2}(1-\delta)\}^{\frac{1}{2}}$ and

$$|c-\tfrac{1}{2}(1-\delta)| < \{F^{-2}-\tfrac{1}{4}(1-\delta^2)\}^{\frac{1}{2}},$$

then $l^2(Y) > 0$ over the subintervals $(-1, -Y'')$ and $(Y'', 1)$ and $l^2(Y) < 0$ elsewhere over the interval of flow. Then subcase (iv) gives

$$\int_{Y''}^1 l\,dY \sim \tfrac{1}{2}m\pi\epsilon \quad \text{as} \quad m \to \infty. \tag{38}$$

Here certain changes in relation (32) have been made on account of the exception noted thereafter. In particular, the n of relation (32) has been doubled to give the correct number m of nodes of the eigenfunctions, which are alternately even and odd functions of Y; also the eigenfunctions here are exponentially small only in the subinterval $(-Y'', Y'')$.

(v) If $|c| < F^{-1}\{2/(1+\delta)\}^{\frac{1}{2}}$ and $F^{-1} < c < 1$ and either (I) $F^{-1} < \frac{1}{2}(1-\delta^2)^{\frac{1}{2}}$ or (II) $F^{-1} > \frac{1}{2}(1-\delta^2)^{\frac{1}{2}}$ and $|c-\frac{1}{2}(1-\delta)| > \{F^{-2}-\frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}$, then $l^2(Y) > 0$ over the subintervals $(-1, -Y'')$, $(-Y', Y')$ and $(Y'', 1)$ and $l^2(Y) < 0$ elsewhere over the interval of flow. The approximate eigenvalue relation for this subcase can be shown to give either

$$\int_0^{Y'} l\,dY \sim n\pi\epsilon \quad \text{or} \quad \int_{Y''}^1 l\,dY \sim n\pi\epsilon \quad \text{as} \quad n \to \infty, \tag{39}$$

through integral values. For the former relation the eigenfunction is exponentially small except in the subinterval $(-Y', Y')$, and for the latter except in the subintervals $(-1, -Y'')$ and $(Y'', 1)$. The number m of nodes changes continuously as

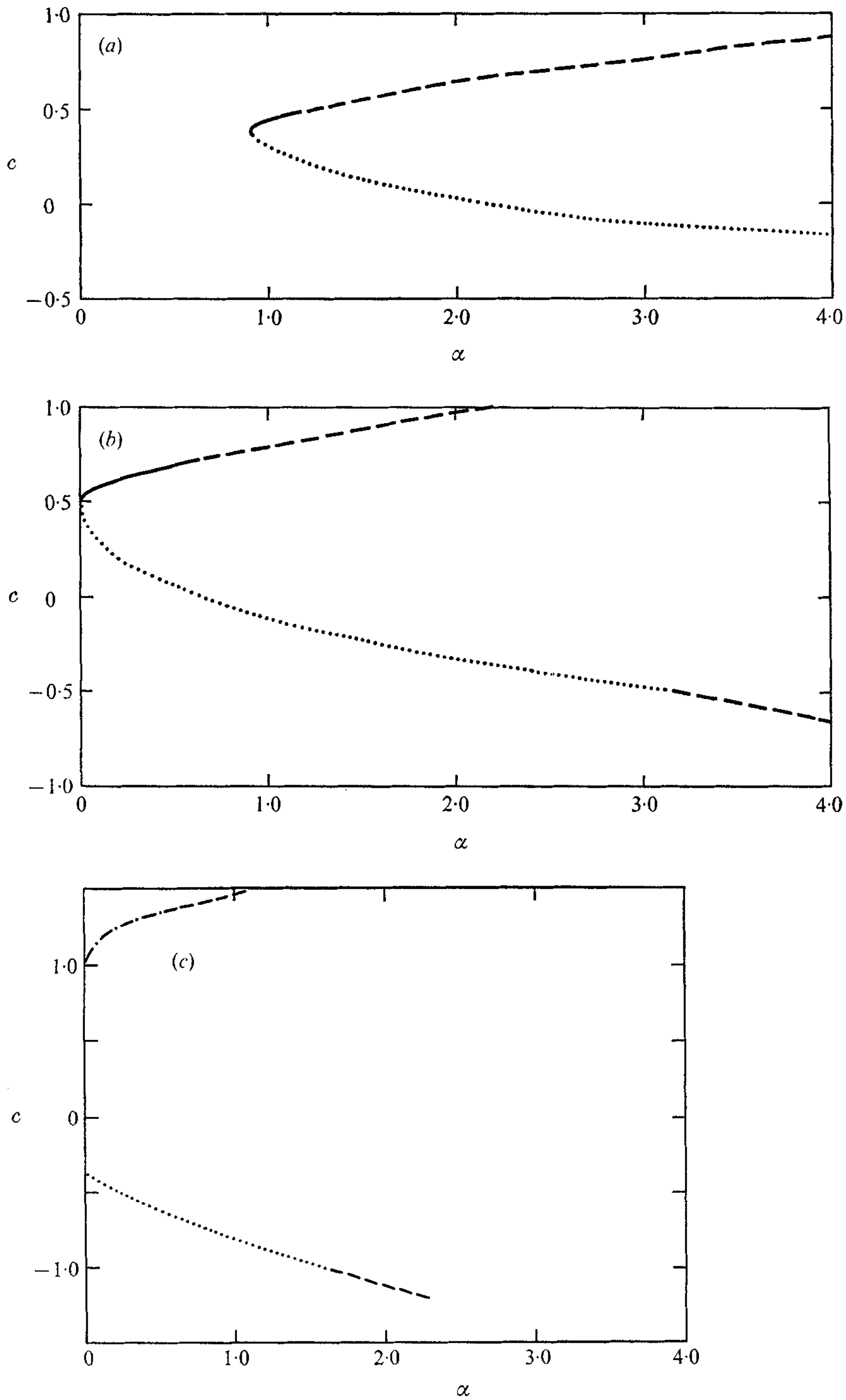

FIGURE 1. Graphs of c against $\alpha = \frac{1}{2}m\pi\epsilon/k$ for $U = 1 - Y^2$, $\delta = 0$. – – –, subcase (i); ······, subcase (iii); – · – · –, subcase (iv); ——, subcase (v). (a) $F = 3$, unstable. (b) $F = 2$, just stable. (c) $F = 1$, stable.

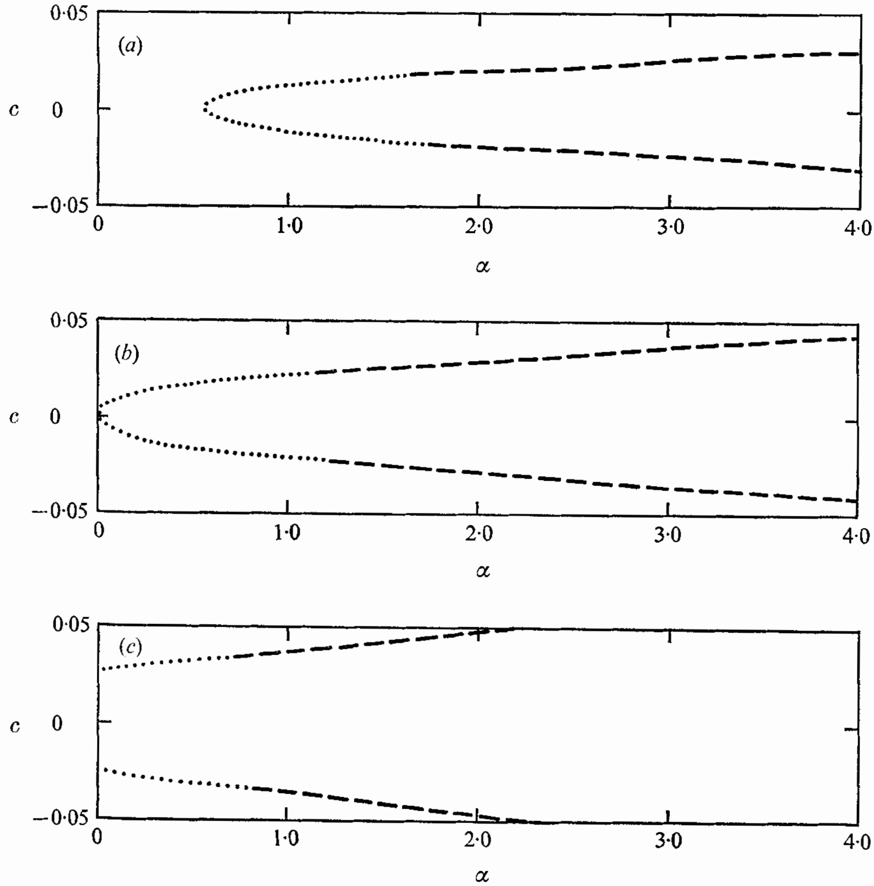

FIGURE 2. Graphs of c against α for $U = 1-Y^2$, $\delta = 0{\cdot}999$. ---, subcase (i); ······, subcase (iii); (a) $F = 60$, unstable. (b) $F = 2/(1-\delta^2)^{\frac{1}{2}} = 44{\cdot}73$, just stable. (c) $F = 30$, stable.

this subcase arises from (i). Also the eigenfunction associated with the former relation (39) has zeros over $(Y'', 1)$, because it oscillates over that subinterval although the coefficient of the oscillating solution is expenentially small.

(b) The unstable modes are difficult to find by the JWKB method because $l^2 = 0$ has eight zeros in the complex Y plane. So we shall have to rest content with the stability criterion and the characteristics of the stable modes already found.

The pattern of the stable modes is illustrated in figures 1 and 2, which are based upon computation of relations (35) and (37)–(39) for various values of the parameters. First note that when c is large, subcase (i) arises and

$$\int_0^1 l\,dY = kF^2\{c^2 - \tfrac{2}{3}(1-\delta)\,c + O(1)\} \quad \text{as} \quad c \to \pm\infty. \tag{40}$$

Accordingly, we have cut off the graphs where formulae (35) and (40) already give the velocities of the stable internal gravity waves moderately accurately.

In figure 1, c is plotted against the quotient $\alpha \equiv \frac{1}{2}m\pi\epsilon/k$ of the average lateral wavenumber and the longitudinal wavenumber for $\delta = 0$ and various values of F.

There is local stability everywhere for all lateral wavenumbers if and only if $F \leqslant 2$. In figure 1 (*a*), $F = 3$ and there are two stable modes when $\frac{1}{2}m\pi\epsilon/k \geqslant 0{\cdot}91$ and the flow is unstable if $\frac{1}{2}m\pi\epsilon/k < 0{\cdot}91$. In figure 1 (*b*), $F = 2$ and the flow is just stable; there are two modes for each pair of real wavenumbers. In figure 1 (*c*), $F = 1$ and the flow is stable; there are two modes for each pair of real wavenumbers, but the values of c between $-\frac{1}{2}(\sqrt{3}-1)$ and 1 do not occur.

We take $\delta = 0{\cdot}999$ in figure 2 to simulate air blowing over water. In figure 2 (*a*), $F = 60$ and the flow is unstable; in figure 2 (*b*), $F = 2/(1-\delta^2)^{\frac{1}{2}} = 44{\cdot}73$ and the flow is just stable; in figure 2(*c*), $F = 30$ and the flow is stable.

6. Broken-line profile of channel flow

The characteristics of unstable modes are easier to find when an explicit exact eigenvalue relation is known, as it is for piecewise linear profiles. So to explore instabilities, we shall represent flow in a channel by the profile

$$U = \begin{cases} 0 & \text{for} \quad \frac{2}{3} < Y \leqslant 1, \\ 1 & \text{for} \quad |Y| < \frac{2}{3}. \end{cases} \tag{41}$$

Therefore

$$l^2 = \begin{cases} l_0^2 \equiv k^2(F^4c^4 - 1) & \text{for} \quad \frac{2}{3} < |Y| \leqslant 1, \\ l_1^2 \equiv k^2F^4\{[c-\frac{1}{2}(1-\delta)]^2 + \frac{1}{4}(1-\delta^2)\}^2 - k^2 & \text{for} \quad |Y| < \frac{2}{3}, \end{cases} \tag{42}$$

and (17) has piecewise sinusoidal solutions. The continuous solution satisfying boundary conditions (18) leads to the eigenvalue relation

$$l_0 T_0 + l_1 T_1 = 0 \tag{43}$$

for an even eigenfunction f, and to

$$l_1 - l_0 T_0 T_1 = 0 \tag{44}$$

for an odd one, where we use the definitions

$$T_0 \equiv \tan \tfrac{1}{3}\epsilon^{-1} l_0, \quad T_1 \equiv \tan \tfrac{2}{3}\epsilon^{-1} l_1. \tag{45}$$

(*a*, i) If either $c > F^{-1}$, or $c < -F^{-1}$ and either (I) $F^{-1} < \frac{1}{2}(1-\delta^2)^{\frac{1}{2}}$ or (II) $|c-\frac{1}{2}(1-\delta)| > \{F^{-2}-\frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}$, then $l_0^2, l_1^2 > 0$ and relations (43) and (44) give eigenvalues which interlace. The relations can be combined to give

$$\tfrac{2}{3}l_0 + \tfrac{4}{3}l_1 \sim m\pi\epsilon \quad \text{as} \quad m \to \infty, \tag{46}$$

consistent with the JWKB approximation (23).

(iii) If $-F^{-1} < c < F^{-1}$ and either

$$\text{(I)} \ F^{-1} < \tfrac{1}{2}(1-\delta^2)^{\frac{1}{2}} \quad \text{or} \quad \text{(II)} \ |c-\tfrac{1}{2}(1-\delta)| > \{F^{-2}-\tfrac{1}{4}(1-\delta^2)\}^{\frac{1}{2}},$$

then $l_0^2 < 0$, $l_1^2 > 0$ and eigenvalue relations (43) and (44) give

$$\tfrac{4}{3}\epsilon^{-1}l_1 \sim m\pi \quad \text{as} \quad m \to \infty. \tag{47}$$

(iv) If either (I) $c > F^{-1}$ or (II) $c < -F^{-1}$ and $|c-\frac{1}{2}(1-\delta)| > \{F^{-2}-\frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}$, then $l_0^2 > 0$, $l_1^2 < 0$ and relations (43) and (44) give

$$\tfrac{2}{3}\epsilon^{-1}l_0 \sim m\pi \quad \text{as} \quad m \to \infty. \tag{48}$$

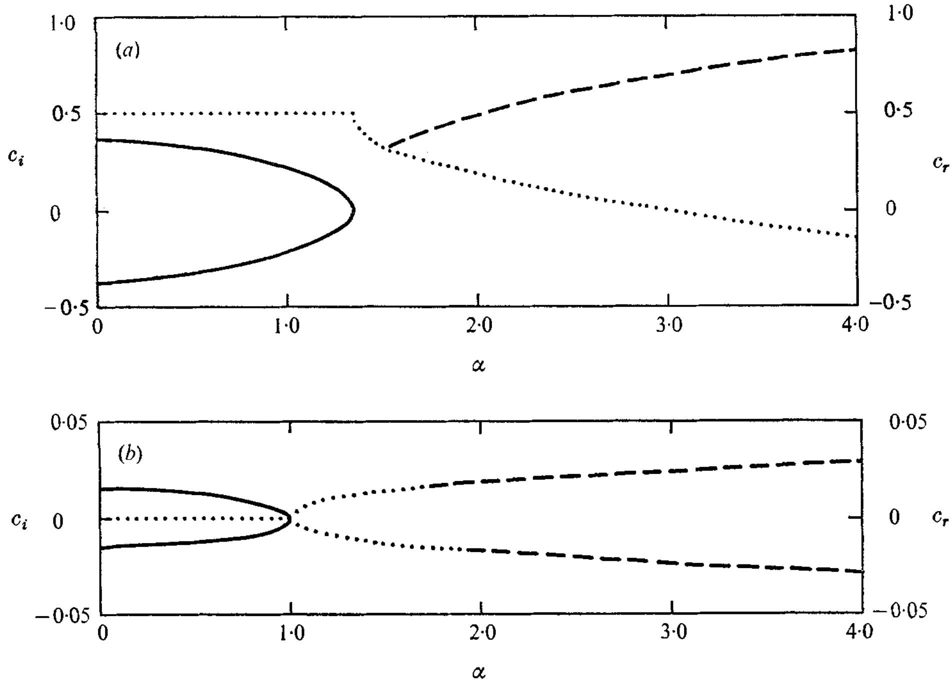

FIGURE 3. Graphs of c against α for broken-line profile (41). ---, subcase (i); ·····, subcase (iii); ——, Im c, given by formula (49). (*a*) $\delta = 0$, $F = 3$. (*b*) $\delta = 0{\cdot}999$, $F = 60$.

(*b*) When the eigenvalue c is complex, one would expect l_0^2 and l_1^2 to be complex in general. However, it is easy to show that relations (43) and (44) cannot be satisfied when both $\mathrm{Im}\,(l_0)/\epsilon$ and $\mathrm{Im}\,(l_1)/\epsilon$ are large. It can further be shown that in fact the JWKB approximations above for real l_0^2 and l_1^2 yield the complex as well as the real eigenvalues c. The complex roots c bifurcate at a critical value of ϵ for each given pair of values of k and m, there being a complex conjugate pair of values of c for ϵ less than, coincident real roots for ϵ equal to and a real pair for ϵ greater than this value. Case (a, iii) above gives bifurcation where

$$\alpha \equiv \tfrac{1}{2}m\pi\epsilon/k = \alpha_c = \tfrac{2}{3}\{\tfrac{1}{16}F^4(1-\delta^2)^2 - 1\}^{\frac{1}{2}},$$

and the eigenvalue

$$c = \tfrac{1}{2}(1-\delta) \pm F^{-1}\{(1+9m^2\pi^2\epsilon^2/16k^2)^{\frac{1}{2}} - \tfrac{1}{4}F^2(1-\delta^2)\}^{\frac{1}{2}}. \qquad (49)$$

Eigenvalue relations are plotted in figure 3. The stable eigenvalues have similar properties to those for the parabolic profile (33) depicted in figures 1 and 2, so we have taken only the cases $\delta = 0$ with $F = 3$ and $\delta = 0{\cdot}999$ with $F = 60$ in figures 3 (*a*) and (*b*), respectively. For the former case, stable modes are given by relations (46) and (47) for $c > F^{-1} = \frac{1}{3}$, $\alpha = (\frac{7}{3})^{\frac{1}{2}}$, and by relation (49) for $\alpha < (\frac{7}{3})^{\frac{1}{2}}$. This gives bifurcation at $\alpha = \alpha_c = 2^{\frac{1}{2}}$, and instability for $\alpha < 2^{\frac{1}{2}}$. For the latter case, stable modes are given by relations (46) and (47) and bifurcation occurs at $c = \frac{1}{2}(1-\delta) = 0{\cdot}0005$, $\alpha = \alpha_c = 1{\cdot}0$.

7. Linear flow

For the last example, take

$$U = Y \quad \text{for} \quad -1 \leqslant Y \leqslant 1. \tag{50}$$

Then the problem (17)–(19) is symmetric up- and downstream, so that if

$$c = c_r + ic_i$$

is an eigenvalue with eigenfunction $f(Y)$ for given values of k, m, F, δ and ϵ then so is $-c^* = -c_r + ic_i$ with $f^*(-Y)$ for the same values. Accordingly, we shall suppose that $c_r \geqslant 0$ without loss of generality. Also the criterion (21) gives local stability everywhere if and only if $F \leqslant 2/(1-\delta^2)^{\frac{1}{2}}$. We shall now proceed in a similar manner to that of §5.

(a, i) If either (I) $c > \{2/(1+\delta)\}^{\frac{1}{2}} F^{-1}$, or (II) $c < \{2/(1+\delta)\}^{\frac{1}{2}} F^{-1}$ and $F < 2/(1-\delta^2)^{\frac{1}{2}}$ and $c < |\frac{1}{2}(1-\delta) - \{F^{-2} - \frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}|$ or (III) $c < \{2/(1+\delta)\}^{\frac{1}{2}} F^{-1}$ and $F < 2/(1-\delta^2)^{\frac{1}{2}}$ and $c > 1$ and $c > \frac{1}{2}(1-\delta) + \{F^{-2} - \frac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}$, then $l^2(Y) > 0$ over the interval $(-1, 1)$ and the eigenvalue relation gives

$$\int_{-1}^{1} l \, dY \sim m\pi\epsilon \quad \text{as} \quad m \to \infty. \tag{51}$$

(ii) If $c < \{2/(1+\delta)\}^{\frac{1}{2}} F^{-1}$ and $F < 2/(1-\delta^2)^{\frac{1}{2}}$ and

$$|\tfrac{1}{2}(1-\delta) - \{F^{-2} - \tfrac{1}{4}(1-\delta^2)\}^{\frac{1}{2}}| < c < \tfrac{1}{2}(1-\delta) + \{F^{-2} - \tfrac{1}{4}(1-\delta^2)\}^{\frac{1}{2}},$$

then $l^2 > 0$ over the subinterval $(-1, Y')$ and $l^2 < 0$ elsewhere over the interval of flow, where Y' and Y'' are defined by

$$Y', Y'' = c \mp [\{2F^{-2} - (1+\delta)c^2\}/(1-\delta)]^{\frac{1}{2}} \tag{52}$$

respectively. The eigenvalue relation then gives

$$\int_{-1}^{Y'} l \, dY \sim m\pi\epsilon \quad \text{as} \quad m \to \infty. \tag{53}$$

(iv) If $c < \{2/(1+\delta)\}^{\frac{1}{2}} F^{-1}$ and either $F > 2/(1-\delta^2)^{\frac{1}{2}}$ or

$$\tfrac{1}{2}(1-\delta) + \{F^{-2} - \tfrac{1}{4}(1-\delta^2)\}^{\frac{1}{2}} < c < 1,$$

then $l^2 > 0$ over the subintervals $(-1, Y')$ and $(Y'', 1)$ and $l^2 < 0$ elsewhere over the interval of flow. It follows that either

$$\int_{-1}^{Y'} l \, dY \quad \text{or} \quad \int_{Y''}^{1} l \, dY \sim n\pi\epsilon \quad \text{as} \quad n \to \infty. \tag{54}$$

The pattern of the stable modes is illustrated in figures 4 and 5, which are based upon computations of relations (51)–(54). Note that

$$\tfrac{1}{2}\pi m\epsilon/k \sim F^2 c^2 \quad \text{as} \quad c \to +\infty, \quad \epsilon \to 0,$$

and that the local criterion of stability (21) is inadequate for this profile when $\delta = 0$.

8. Conclusions

We have considered in detail the Kelvin–Helmholtz instability of some slowly varying flows, instability that has important applications in oceanography and meteorology. However, the ideas and techniques may be used more widely for

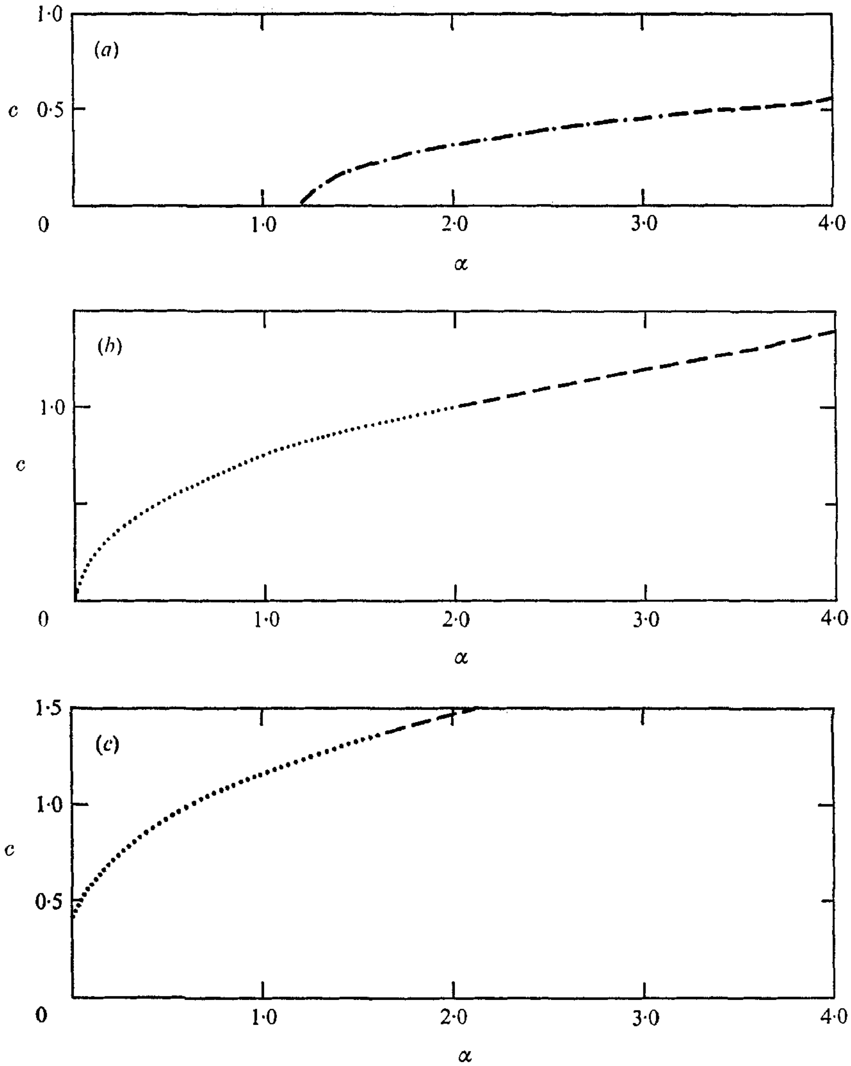

FIGURE 4. Graphs of c against α for $U = Y$, $\delta = 0$. ---, subcase (i); ⋯⋯, subcase (iii); –·–·–, subcase (iv). (*a*) $F = 3$, unstable. (*b*) $F = 2^{\frac{1}{2}}$, just stable. (*c*) $F = 1$, stable.

instability of other slowly varying flows, so our conclusions should be interpreted accordingly.

The chief conclusion has been to substantiate in part the intuition that if the local criterion (21) for stability is satisfied *everywhere* in the domain of flow, then the given modes will be stable, but if it is violated *anywhere* they will be unstable. However, the intuition was found false for the linear profile of § 7.

Any given linear mode has the same relative growth rate kc at each point, whether it is stable or unstable, but its spatial structure is more complicated. A stable mode may be exponentially small in certain regions, where its local lateral wavenumber l is pure imaginary, but oscillate sinusoidally elsewhere. This affords a mechanism for enhanced response of a stable wave forced by a

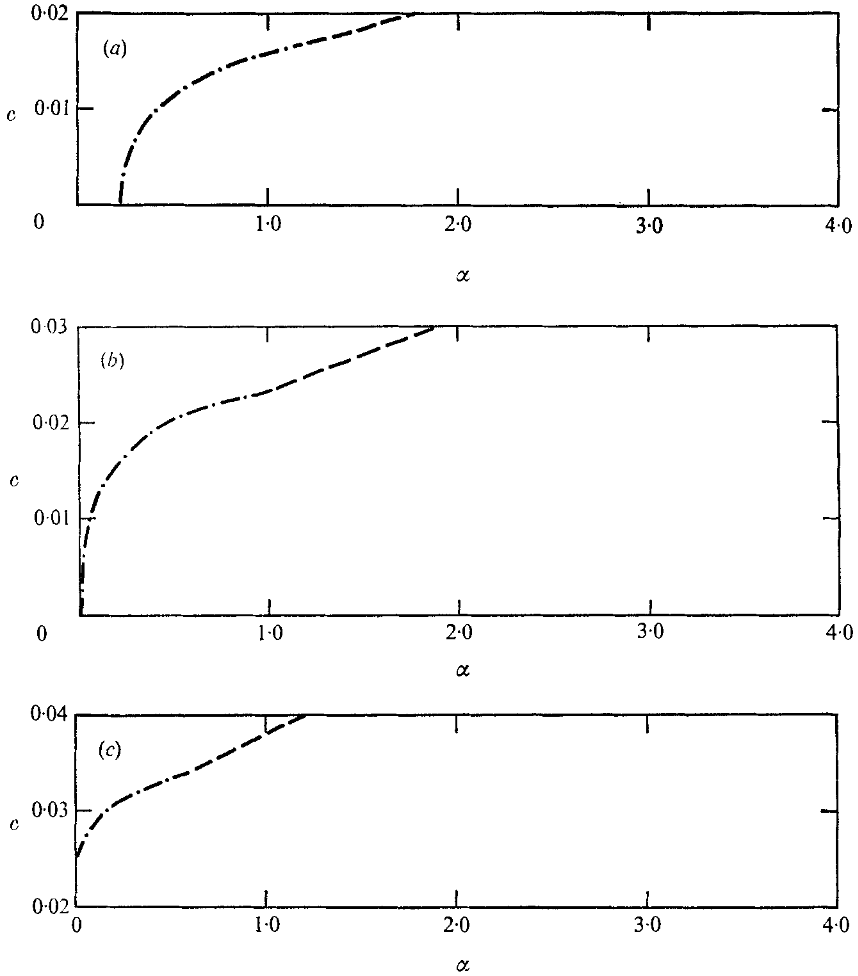

FIGURE 5. Graphs of c against α for $U = Y$, $\delta = 0{\cdot}999$. ---, subcase (i); –·–·–, subcase (iv). (a) $F = 60$, unstable. (b) $F = 44{\cdot}73$, just stable. (c) $F = 30$, stable.

source in a region where the wavenumber is imaginary. An unstable mode may be exponentially damped, oscillate sinusoidally or have a damped oscillation as y varies, according to whether the lateral wavenumber is locally pure imaginary, real or complex respectively.

It may be helpful to regard the lateral wavenumber of an unstable mode as real where the mode is locally stable and complex elsewhere, so that the unstable mode is exponentially small where the flow is locally stable and sinusoidal elsewhere. This is an accurate description of the instabilities of the broken-line profile of §6, but in general is an over-simplified view. However, it gives one intuition into the generation of waves on a calm lake by a gust of wind: the lake surface is ruffled in a patch where the wind is strong enough to cause local instability with modes varying sinusoidally in space, and the lake surface seems calm elsewhere, where the modes decay exponentially in space. (Of course, the linear model here with a steady basic wind independent of height is not strictly applicable to the effects of a gust of wind on a lake, but the essence of the intuition is.) This gives a picture of an unstable mode at each instant as exponentially small where the wind is weak and sinusoidal where the wind is strong, the picture of a wave function of a bound particle in a potential well. In addition, the unstable mode everywhere grows exponentially in time until nonlinear effects occur.

The mechanism of instability is the one discovered by Kelvin, but the strength of the mechanism varies with the lateral co-ordinate y.

The detailed stability characteristics for parabolic, piecewise constant and linear profiles across the channel are given in §§5–7. These make a quantitative comparison of experiments and theory of Kelvin–Helmholtz instability a step nearer. A channel in the laboratory necessarily has a finite width, which is ignored in the classical theory of Kelvin. However, we have ignored viscosity and use an over-simplified vertical dependence of the basic velocity.

We have chosen the basic flow (1) both for its geophysical applications and for its relative simplicity. It is not difficult to allow for a slowly varying velocity in the lower fluid and for interfacial surface tension; this leads to problems of the same form but with greater algebraic complications in the relationship of l to c. One may hope similarly to allow the basic flow to vary slowly with x instead of y. The more challenging problem in which the basic flow varies slowly with both x and y would seem to demand the method of ray theory or some other appropriate generalization of the JWKB method.

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